Lie Groups for Beginners

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March 1, 2010

1 Basic Lie Group Concepts

1.1 A Manifold and a Group

A Lie group G is a manifold that possesses a smooth group operation. Associated with it is a Lie Algebra $\mathfrak{g}$ which, loosely speaking, can be identified with the tangent space at the identity and completely defines how the groups behaves around the identity. There is a mapping from $\mathfrak{g}$ back to G, called the exponential map

$$\exp : \mathfrak{g} \rightarrow G$$

and a corresponding inverse

$$\log : G \rightarrow \mathfrak{g}$$

that maps elements in G to an element in $\mathfrak{g}$.

1.2 Lie Algebra

The Lie Algebra $\mathfrak{g}$ is called an algebra because it is endowed with a binary operation, the Lie bracket $[X,Y]$, the properties of which are closely related to the group operation of G. For example, in matrix Lie groups, the Lie bracket is given by $[A,B] \stackrel{\Delta}{=} AB - BA$. The relationship with the group operation is as follows: for commutative Lie groups vector addition $X+Y$ in $\mathfrak{g}$ mimicks the group operation. For example, if we have $Z = X+Y$ in $\mathfrak{g}$, when mapped backed to G via the exponential map we obtain

$$e^Z = e^{X+Y} = e^X e^Y$$

However, this does *not* hold for non-commutative Lie groups:

$$Z = \log(e^X e^Y) \neq X+Y$$

Instead, Z can be calculated using the Baker-Campbell-Hausdorff (BCH) formula:[1]

$$Z = X + Y + [X,Y]/2 + [X-Y,[X,Y]]/12 - [Y,[X,[X,Y]]]/24 + \ldots$$

For commutative groups the bracket is zero and we recover $Z = X+Y$. For non-commutative groups we can use the BCH formula to approximate it.

1.3 Exponential Coordinates

For n-dimensional matrix Lie groups, the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{R}^n$, and we can define the mapping

$$\hat{} : \mathbb{R}^n \rightarrow \mathfrak{g}$$

$$\hat{} : x \rightarrow \hat{x}$$

which maps n-vectors $x \in \mathbb{R}^n$ to elements of $\mathfrak{g}$. In the case of matrix Lie groups, the elements $\hat{x}$ of $\mathfrak{g}$ are $n \times n$ matrices, and the map is given by

$$\hat{x} = \sum_{i=1}^{n} x_i G^i \tag{1}$$

[1] http://en.wikipedia.org/wiki/Baker–Campbell–Hausdorff_formula

where the G^i are $n \times n$ matrices known as the Lie group generators. The meaning of the map $x \to \hat{x}$ will depend on the group G and will be very intuitive.

1.4 The Adjoint Map

Below we frequently make use of the equality[2]

$$g e^{\hat{x}} g^{-1} = e^{Ad_g \hat{x}}$$

where $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is a map parameterized by a group element g. The intuitive explanation is that a change $\exp(\hat{x})$ defined around the orgin, but applied at the group element g, can be written in one step by taking the adjoint $Ad_g \hat{x}$ of $\hat{x}$. In the case of a matrix group the ajoint can be written as [3]

$$Ad_T \hat{x} \stackrel{\Delta}{=} T e^{\hat{x}} T^{-1}$$

and hence we have

$$T e^{\hat{x}} T^{-1} = e^{T \hat{x} T^{-1}}$$

where both T and $\hat{x}$ are $n \times n$ matrices for an n-dimensional Lie group.

1.5 Actions

The (usual) action of an n-dimensional matrix group G is matrix-vector multiplication on $\mathbb{R}^n$,

$$q = Tp$$

with $p, q \in \mathbb{R}^n$ and $T \in GL(n)$.

[2] http://en.wikipedia.org/wiki/Exponential_map

[3] http://en.wikipedia.org/wiki/Adjoint_representation_of_a_Lie_group

2 2D Rotations

We first look at a very simple group, the 2D rotations.

2.1 Basics

The Lie group $SO(2)$ is a subgroup of the general linear group $GL(2)$ of 2×2 invertible matrices. Its Lie algebra $\mathfrak{so}(2)$ is the vector space of 2×2 skew-symmetric matrices. Since $SO(2)$ is a one-dimensional manifold, $\mathfrak{so}(2)$ is isomorphic to $\mathbb{R}$ and we define

$$\hat{}: \mathbb{R} \to \mathfrak{so}(2)$$

$$\hat{}: \theta \to \hat{\theta} = [\theta]_+$$

which maps the angle θ to the 2×2 skew-symmetric matrix $[\theta]_+$:

$$[\theta]_+ = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$$

The exponential map can be computed in closed form as

$$R = e^{[\theta]_+} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2.2 Actions

In the case of $SO(2)$ the vector space is $\mathbb{R}^2$, and the group action corresponds to rotating a point

$$q = Rp$$

We would now like to know what an incremental rotation parameterized by θ would do:

$$q(\theta) = Re^{[\theta]_+}p$$

hence the derivative is:

$$\frac{\partial q(\omega)}{\partial \omega} = R\frac{\partial}{\partial \omega}\left(e^{[\theta]_+}p\right) = R\frac{\partial}{\partial \omega}\left([\theta]_+ p\right) = RH_p$$

Note that

$$[\theta]_+ \begin{bmatrix} x \\ y \end{bmatrix} = \theta R_{\pi/2} \begin{bmatrix} x \\ y \end{bmatrix} = \theta \begin{bmatrix} -y \\ x \end{bmatrix} \quad (2)$$

which acts like a restricted “cross product” in the plane.

3 2D Rigid Transformations

3.1 Basics

The Lie group $SE(2)$ is a subgroup of the general linear group $GL(3)$ of 3×3 invertible matrices of the form

$$T \stackrel{\Delta}{=} \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

where $R \in SO(2)$ is a rotation matrix and $t \in \mathbb{R}^2$ is a translation vector. Its Lie algebra $\mathfrak{se}(2)$ is the vector space of 3×3 twists $\hat{\xi}$ parameterized by the *twist coordinates* $\xi \in \mathbb{R}^3$, with the mapping

$$\xi \stackrel{\Delta}{=} \begin{bmatrix} v \\ \omega \end{bmatrix} \rightarrow \hat{\xi} \stackrel{\Delta}{=} \begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix}$$

Note we think of robots as having a pose (x, y, θ) and hence I reserved the first two components for translation and the last for rotation. The Lie group generators are

$$G^x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; G^y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \; G^\theta = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying the exponential map to a twist ξ yields a screw motion yielding an element in $SE(2)$:

$$T = \exp \hat{\xi}$$

A closed form solution for the exponential map is in the works...

3.2 The Adjoint Map

The adjoint is

$$\begin{aligned} Ad_T \hat{\xi} &= T \hat{\xi} T^{-1} \\ &= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} [\omega]_+ & -[\omega]_+ t + Rv \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} [\omega]_+ & Rv - \omega R_{\pi/2} t \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{3}$$

From this we can express the Adjoint map in terms of plane twist coordinates:

$$\begin{bmatrix} v' \\ \omega' \end{bmatrix} = \begin{bmatrix} R & -R_{\pi/2} t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

3.3 Actions

The action of $SE(2)$ on 2D points is done by embedding the points in $\mathbb{R}^3$ by using homogeneous coordinates

$$\hat{q} = \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = T\hat{p}$$

Analoguous to $SE(3)$, we can compute a velocity $\hat{\xi}\hat{p}$ in the local T frame:

$$\hat{\xi}\hat{p} = \begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} [\omega]_+ p + v \\ 0 \end{bmatrix}$$

By only taking the top two rows, we can write this as a velocity in $\mathbb{R}^2$, as the product of a 2×3 matrix H_p that acts upon the exponential coordinates ξ directly:

$$[\omega]_+ p + v = v + R_{\pi/2} p \omega = \begin{bmatrix} I_2 & R_{\pi/2} p \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} = H_p \xi$$

4 3D Rotations

4.1 Basics

The Lie group $SO(3)$ is a subgroup of the general linear group $GL(3)$ of 3×3 invertible matrices. Its Lie algebra $\mathfrak{so}(3)$ is the vector space of 3×3 skew-symmetric matrices. The exponential map can be computed in closed form using Rodrigues' formula.

Since $SO(3)$ is a three-dimensional manifold, $\mathfrak{so}(3)$ is isomorphic to $\mathbb{R}^3$ and we define the map

$$\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$$

$$\hat{}: \omega \to \hat{\omega} = [\omega]_\times$$

which maps 3-vectors ω to skew-symmetric matrices $[\omega]_\times$:

$$[\omega]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \omega_x G^x + \omega_y G^y + \omega_z G^z$$

where the G^i are the generators for $SO(3)$,

$$G^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} G^y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} G^z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

corresponding to a rotation around X, Y, and Z, respectively. The Lie bracket $[x, y]$ corresponds to the cross product $x \times y$ in $\mathbb{R}^3$.

For every $3-$vector ω there is a corresponding rotation matrix

$$R = e^{[\omega]_\times}$$

and this is defines the canonical parameterization of $SO(3)$, with ω known as the canonical or exponential coordinates. It is equivalent to the axis-angle representation for rotations, where the unit vector $\omega / \|\omega\|$ defines the rotation axis, and its magnitude the amount of rotation θ.

4.2 The Adjoint Map

For rotation matrices R we can prove the following identity (see 5 on page 11):

$$R[\omega]_\times R^T = [R\omega]_\times \tag{4}$$

Hence, given property (5), the adjoint map for $\mathfrak{so}(3)$ simplifies to

$$Ad_R[\omega]_\times = R[\omega]_\times R^T = [R\omega]_\times$$

and this can be expressed in exponential coordinates simply by rotating the axis ω to $R\omega$.

As an example, to apply an axis-angle rotation ω to a point p in the frame R, we could:

1. First transform p back to the world frame, apply ω, and then rotate back:

$$q = Re^{[\omega]_\times} R^T$$

2. Immediately apply the transformed axis-angle transformation $Ad_R[\omega]_\times = [R\omega]_\times$:

$$q = e^{[R\omega]_\times} p$$

4.3 Actions

In the case of $SO(3)$ the vector space is $\mathbb{R}^3$, and the group action corresponds to rotating a point

$$q = Rp$$

We would now like to know what an incremental rotation parameterized by ω would do:

$$q(\omega) = Re^{[\omega]_\times} p$$

hence the derivative is:

$$\frac{\partial q(\omega)}{\partial \omega} = R\frac{\partial}{\partial \omega}\left(e^{[\omega]_\times} p\right) = R\frac{\partial}{\partial \omega}\left([\omega]_\times p\right) = RH_p$$

To calculate H_p we make use of

$$[\omega]_\times p = \omega \times p = -p \times \omega = [-p]_\times \omega$$

5 3D Rigid Transformations

The Lie group $SE(3)$ is a subgroup of the general linear group $GL(4)$ of 4×4 invertible matrices of the form

$$T \stackrel{\Delta}{=} \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

where $R \in SO(3)$ is a rotation matrix and $t \in \mathbb{R}^3$ is a translation vector. Its Lie algebra $\mathfrak{se}(3)$ is the vector space of 4×4 twists $\hat{\xi}$ parameterized by the *twist coordinates* $\xi \in \mathbb{R}^6$, with the mapping [1]

$$\xi \stackrel{\Delta}{=} \begin{bmatrix} \omega \\ v \end{bmatrix} \rightarrow \hat{\xi} \stackrel{\Delta}{=} \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix}$$

Note we follow Frank Park's convention and reserve the first three components for rotation, and the last three for translation. Hence, with this parameterization, the generators for $SE(3)$ are

$$G^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G^5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G^6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Applying the exponential map to a twist ξ yields a screw motion yielding an element in $SE(3)$:

$$T = \exp \hat{\xi}$$

A closed form solution for the exponential map is given in [1, page 42].

5.1 The Adjoint Map

The adjoint is

$$\begin{aligned} Ad_T \hat{\xi} &= T \hat{\xi} T^{-1} \\ &= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} [R\omega]_\times & -[R\omega]_\times t + Rv \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} [R\omega]_\times & t \times R\omega + Rv \\ 0 & 0 \end{bmatrix} \end{aligned}$$

From this we can express the Adjoint map in terms of twist coordinates (see also [1] and FP):

$$\begin{bmatrix} \omega' \\ v' \end{bmatrix} = \begin{bmatrix} R & 0 \\ [t]_\times R & R \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix}$$

5.2 Actions

The action of $SE(3)$ on 3D points is done by embedding the points in $\mathbb{R}^4$ by using homogeneous coordinates

$$\hat{q} = \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = T\hat{p}$$

We would now like to know what an incremental rotation parameterized by ξ would do:

$$\hat{q}(\xi) = Te^{\hat{\xi}}\hat{p}$$

hence the derivative (following the exposition in Section **??**):

$$\frac{\partial \hat{q}(\xi)}{\partial \xi} = T\frac{\partial}{\partial \xi}\left(\hat{\xi}\hat{p}\right) = TH_p$$

where $\hat{\xi}\hat{p}$ corresponds to a velocity in $\mathbb{R}^4$ (in the local T frame):

$$\hat{\xi}\hat{p} = \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} \omega \times p + v \\ 0 \end{bmatrix}$$

Notice how velocities are anologous to points at infinity in projective geometry: they correspond to free vectors indicating a direction and magnitude of change.

By only taking the top three rows, we can write this as a velocity in $\mathbb{R}^3$, as the product of a 3×6 matrix H_p that acts upon the exponential coordinates ξ directly:

$$\omega \times p + v = -p \times \omega + v = \begin{bmatrix} -[p]_\times & I_3 \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} = H_p\xi$$

Appendix: Proof of Property 5

We can prove the following identity for rotation matrices R,

$$\begin{aligned}
R[\omega]_\times R^T &= R[\omega]_\times \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \\
&= R \begin{bmatrix} \omega \times a_1 & \omega \times a_2 & \omega \times a_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1(\omega \times a_1) & a_1(\omega \times a_2) & a_1(\omega \times a_3) \\ a_2(\omega \times a_1) & a_2(\omega \times a_2) & a_2(\omega \times a_3) \\ a_3(\omega \times a_1) & a_3(\omega \times a_2) & a_3(\omega \times a_3) \end{bmatrix} \\
&= \begin{bmatrix} \omega(a_1 \times a_1) & \omega(a_2 \times a_1) & \omega(a_3 \times a_1) \\ \omega(a_1 \times a_2) & \omega(a_2 \times a_2) & \omega(a_3 \times a_2) \\ \omega(a_1 \times a_3) & \omega(a_2 \times a_3) & \omega(a_3 \times a_3) \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\omega a_3 & \omega a_2 \\ \omega a_3 & 0 & -\omega a_1 \\ -\omega a_2 & \omega a_1 & 0 \end{bmatrix} \\
&= [R\omega]_\times \qquad (5)
\end{aligned}$$

where a_1, a_2, and a_3 are the *rows* of R. Above we made use of the orthogonality of rotation matrices and the triple product rule:

$$a(b \times c) = b(c \times a) = c(a \times b)$$

References

[1] R.M. Murray, Z. Li, and S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.