Geometry Derivatives and Other Hairy Math

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1 Review of Lie Groups

1.1 A Manifold and a Group

A Lie group G is a manifold that possesses a smooth group operation. Associated with it is a Lie Algebra $\mathfrak{g}$ which, loosely speaking, can be identified with the tangent space at the identity and completely defines how the groups behaves around the identity. There is a mapping from $\mathfrak{g}$ back to G, called the exponential map

$$\exp : \mathfrak{g} \rightarrow G$$

and a corresponding inverse

$$\log : G \rightarrow \mathfrak{g}$$

that maps elements in G to an element in $\mathfrak{g}$.

1.2 Lie Algebra

The Lie Algebra $\mathfrak{g}$ is called an algebra because it is endowed with a binary operation, the Lie bracket $[X,Y]$, the properties of which are closely related to the group operation of G. For example, in matrix Lie groups, the Lie bracket is given by $[A,B] \stackrel{\Delta}{=} AB - BA$. The Lie bracket does not mimick the group operation, as in non-commutative Lie groups we do not have the usual simplification

$$e^Z = e^X e^Y \neq e^{X+Y}$$

where X, Y, and Z elements of the Lie algebra $\mathfrak{g}$. Instead, Z can be calculated using the Baker-Campbell-Hausdorff (BCH) formula:[1]

$$Z = X + Y + [X,Y]/2 + [X - Y,[X,Y]]/12 - [Y,[X,[X,Y]]]/24 + \ldots$$

For commutative groups the bracket is zero and we recover $Z = X + Y$. For non-commutative groups we can use the BCH formula to approximate it.

[1] http://en.wikipedia.org/wiki/Baker–Campbell–Hausdorff_formula

1.3 Exponential Coordinates

For n-dimensional matrix Lie groups, the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{R}^n$, and we can define the mapping

$$\hat{}:\mathbb{R}^n \to \mathfrak{g}$$

$$\hat{}:x \to \hat{x}$$

which maps n-vectors $x \in \mathbb{R}^n$ to elements of $\mathfrak{g}$. In the case of matrix Lie groups, the elements $\hat{x}$ of $\mathfrak{g}$ are $n \times n$ matrices, and the map is given by

$$\hat{x} = \sum_{i=1}^{n} x_i G^i \tag{1}$$

where the G^i are $n \times n$ matrices known as the Lie group generators. The meaning of the map $x \to \hat{x}$ will depend on the group G and will be very intuitive.

1.4 The Adjoint Map

Below we frequently make use of the equality[2]

$$g e^{\hat{x}} g^{-1} = e^{Ad_g \hat{x}}$$

where $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is a map parameterized by a group element g. The intuitive explanation is that a change $\exp(\hat{x})$ defined around the orgin, but applied at the group element g, can be written in one step by taking the adjoint $Ad_g \hat{x}$ of $\hat{x}$. In the case of a matrix group the ajoint can be written as [3]

$$Ad_T \hat{x} \stackrel{\Delta}{=} T e^{\hat{x}} T^{-1}$$

and hence we have

$$T e^{\hat{x}} T^{-1} = e^{T \hat{x} T^{-1}}$$

where both T and $\hat{x}$ are $n \times n$ matrices for an n-dimensional Lie group. Below we introduce the most important Lie groups that we deal with.

2 Derivatives of Mappings

The derivatives for *inverse, compose*, and *between* can be derived from Lie group principles. Specifically, to find the derivative of a function $f(g)$, we want to find the Lie algebra element $\hat{y} \in \mathfrak{g}$, that will result from changing g using $\hat{x}$, also in exponential coordinates:

$$f(g)\, e^{\hat{y}} = f\left(g e^{\hat{x}}\right)$$

Calculating these derivatives requires that we know the form of the function f.

[2] http://en.wikipedia.org/wiki/Exponential_map
[3] http://en.wikipedia.org/wiki/Adjoint_representation_of_a_Lie_group

Starting with **inverse**, i.e., $f(g) = g^{-1}$, we have

$$\begin{aligned} g^{-1}e^{\hat{y}} &= \left(ge^{\hat{x}}\right)^{-1} = e^{-\hat{x}}g^{-1} \\ e^{\hat{y}} &= ge^{-\hat{x}}g^{-1} = e^{Ad_g(-\hat{x})} \\ \hat{y} &= Ad_g\left(-\hat{x}\right) \end{aligned} \tag{2}$$

In other words, and this is very intuitive in hindsight, the inverse is just negation of $\hat{x}$, along with an adjoint to make sure it is applied in the right frame!

Compose can be derived similarly. Let us define two functions to find the derivatives in first and second arguments:

$$f_1(g) = gh \text{ and } f_2(h) = gh$$

The latter is easiest, as a change $\hat{x}$ in the second argument h simply gets applied to the result gh:

$$\begin{aligned} f_2(h)e^{\hat{y}} &= f_2\left(he^{\hat{x}}\right) \\ ghe^{\hat{y}} &= ghe^{\hat{x}} \\ \hat{y} &= \hat{x} \end{aligned} \tag{3}$$

The derivative for the first argument is a bit trickier:

$$\begin{aligned} f_1(g)e^{\hat{y}} &= f_1\left(ge^{\hat{x}}\right) \\ ghe^{\hat{y}} &= ge^{\hat{x}}h \\ e^{\hat{y}} &= h^{-1}e^{\hat{x}}h = e^{Ad_{h^{-1}}\hat{x}} \\ \hat{y} &= Ad_{h^{-1}}\hat{x} \end{aligned} \tag{4}$$

In other words, to apply a change $\hat{x}$ in g we first need to undo h, then apply $\hat{x}$, and then apply h again. All can be done in one step by simply applying $Ad_{h^{-1}}\hat{x}$.

Finally, let us find the derivative of **between**, defined as $between(g,h) = compose(inverse(g),h)$. The derivative in the second argument h is similarly trivial: $\hat{y} = \hat{x}$. The first argument goes as follows:

$$\begin{aligned} f_1(g)e^{\hat{y}} &= f_1\left(ge^{\hat{x}}\right) \\ g^{-1}he^{\hat{y}} &= \left(ge^{\hat{x}}\right)^{-1}h = e^{(-\hat{x})}g^{-1}h \\ e^{\hat{y}} &= \left(h^{-1}g\right)e^{(-\hat{x})}\left(h^{-1}g\right)^{-1} = e^{Ad_{\left(h^{-1}g\right)}(-\hat{x})} \\ \hat{y} &= Ad_{\left(h^{-1}g\right)}\left(-\hat{x}\right) = Ad_{between(h,g)}\left(-\hat{x}\right) \end{aligned} \tag{5}$$

Hence, now we undo h and then apply the inverse $(-\hat{x})$ in the g frame.

3 Derivatives of Actions

The (usual) action of an n-dimensional matrix group G is matrix-vector multiplication on $\mathbb{R}^n$,

$$q = Tp$$

with $p, q \in \mathbb{R}^n$ and $T \in GL(n)$. Let us first do away with the derivative in p, which is easy:

$$\frac{\partial (Tp)}{\partial p} = T$$

We would now like to know what an incremental action $\hat{x}$ would do, through the exponential map

$$q(x) = Te^{\hat{x}}p$$

with derivative

$$\frac{\partial q(x)}{\partial x} = T\frac{\partial}{\partial x}\left(e^{\hat{x}}p\right)$$

Since the matrix exponential is given by the series

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

we have, to first order

$$e^{\hat{x}}p = p + \hat{x}p + \ldots$$

and the derivative of an incremental action x at the origin is

$$H_p \triangleq \frac{\partial e^{\hat{x}}p}{\partial x} = \frac{\partial (\hat{x}p)}{\partial x}$$

Recalling the definition (1) of the map $x \to \hat{x}$, we can calculate $\hat{x}p$ as (using tensor notation)

$$(\hat{x}p)_{jk} = G^i_{jk}x_ip^k$$

and hence the derivative is

$$(H_p)^i_j = G^i_{jk}p^k$$

and the final derivative becomes

$$\frac{\partial q(x)}{\partial x} = TH_p$$

4 3D Rotations

4.1 Basics

The Lie group $SO(3)$ is a subgroup of the general linear group $GL(3)$ of 3×3 invertible matrices. Its Lie algebra $\mathfrak{so}(3)$ is the vector space of 3×3 skew-symmetric matrices. The exponential map can be computed in closed form using Rodrigues' formula.

Since $SO(3)$ is a three-dimensional manifold, $\mathfrak{so}(3)$ is isomorphic to $\mathbb{R}^3$ and we define the map

$$\hat{} : \mathbb{R}^3 \to \mathfrak{so}(3)$$

$$\hat{} : \omega \to \hat{\omega} = [\omega]_\times$$

which maps 3-vectors ω to skew-symmetric matrices $[\omega]_\times$:

$$[\omega]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \omega_x G^x + \omega_y G^y + \omega_z G^z$$

where the G^i are the generators for $SO(3)$,

$$G^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} G^y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} G^z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

corresponding to a rotation around X, Y, and Z, respectively. The Lie bracket $[x, y]$ corresponds to the cross product $x \times y$ in $\mathbb{R}^3$.

For every $3-$vector ω there is a corresponding rotation matrix

$$R = e^{[\omega]_\times}$$

and this is defines the canonical parameterization of $SO(3)$, with ω known as the canonical or exponential coordinates. It is equivalent to the axis-angle representation for rotations, where the unit vector $\omega / \|\omega\|$ defines the rotation axis, and its magnitude the amount of rotation θ.

4.2 The Adjoint Map

We can prove the following identity for rotation matrices R,

$$\begin{aligned} R[\omega]_\times R^T &= R[\omega]_\times \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \\ &= R \begin{bmatrix} \omega \times a_1 & \omega \times a_2 & \omega \times a_3 \end{bmatrix} \\ &= \begin{bmatrix} a_1(\omega \times a_1) & a_1(\omega \times a_2) & a_1(\omega \times a_3) \\ a_2(\omega \times a_1) & a_2(\omega \times a_2) & a_2(\omega \times a_3) \\ a_3(\omega \times a_1) & a_3(\omega \times a_2) & a_3(\omega \times a_3) \end{bmatrix} \\ &= \begin{bmatrix} \omega(a_1 \times a_1) & \omega(a_2 \times a_1) & \omega(a_3 \times a_1) \\ \omega(a_1 \times a_2) & \omega(a_2 \times a_2) & \omega(a_3 \times a_2) \\ \omega(a_1 \times a_3) & \omega(a_2 \times a_3) & \omega(a_3 \times a_3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\omega a_3 & \omega a_2 \\ \omega a_3 & 0 & -\omega a_1 \\ -\omega a_2 & \omega a_1 & 0 \end{bmatrix} \\ &= [R\omega]_\times \end{aligned} \tag{6}$$

where a_1, a_2, and a_3 are the *rows* of R. Above we made use of the orthogonality of rotation matrices and the triple product rule:

$$a(b \times c) = b(c \times a) = c(a \times b)$$

Hence, given property (6), the adjoint map for $\mathfrak{so}(3)$ simplifies to

$$Ad_R[\omega]_\times = R[\omega]_\times R^T = [R\omega]_\times$$

and this can be expressed in exponential coordinates simply by rotating the axis ω to $R\omega$.

As an example, to apply an axis-angle rotation ω to a point p in the frame R, we could:

1. First transform p back to the world frame, apply ω, and then rotate back:

$$q = Re^{[\omega]_\times} R^T$$

2. Immediately apply the transformed axis-angle transformation $Ad_R[\omega]_\times = [R\omega]_\times$:

$$q = e^{[R\omega]_\times} p$$

4.3 Derivatives of Mappings

Hence, we are now in a position to simply posit the derivative of **inverse**,

$$\begin{aligned} [\omega']_\times &= Ad_R\left([-\omega]_\times\right) = [R(-\omega)]_\times \\ \frac{\partial R^T}{\partial \omega} &= -R \end{aligned}$$

compose in its first argument,

$$\begin{aligned} [\omega']_\times &= Ad_{R_2^T}\left([\omega]_\times\right) = [R_2^T \omega]_\times \\ \frac{\partial \left(R_1 R_2\right)}{\partial \omega_1} &= R_2^T \end{aligned}$$

compose in its second argument,

$$\frac{\partial \left(R_1 R_2\right)}{\partial \omega_2} = I_3$$

between in its first argument,

$$\begin{aligned} [\omega']_\times &= Ad_{R_2^T R_1}\left([-\omega]_\times\right) = [R_2^T R_1(-\omega)]_\times \\ \frac{\partial \left(R_1^T R_2\right)}{\partial \omega_1} &= -R_2^T R_1 = -between(R_2, R_1) \end{aligned}$$

and between in its second argument,

$$\frac{\partial \left(R_1^T R_2\right)}{\partial \omega_2} = I_3$$

4.4 Derivatives of Actions

In the case of $SO(3)$ the vector space is $\mathbb{R}^3$, and the group action corresponds to rotating a point

$$q = Rp$$

We would now like to know what an incremental rotation parameterized by ω would do:

$$q(\omega) = Re^{[\omega]_\times} p$$

hence the derivative (following the exposition in Section 3):

$$\frac{\partial q(\omega)}{\partial \omega} = R\frac{\partial}{\partial \omega}\left(e^{[\omega]_\times} p\right) = R\frac{\partial}{\partial \omega}\left([\omega]_\times p\right) = RH_p$$

To calculate H_p we make use of

$$[\omega]_\times p = \omega \times p = -p \times \omega = [-p]_\times \omega$$

Hence, the final derivative of an action in its first argument is

$$\frac{\partial q(\omega)}{\partial \omega} = RH_p = R[-p]_\times$$

5 3D Rigid Transformations

The Lie group $SE(3)$ is a subgroup of the general linear group $GL(4)$ of 4×4 invertible matrices of the form

$$T \stackrel{\Delta}{=} \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

where $R \in SO(3)$ is a rotation matrix and $t \in \mathbb{R}^3$ is a translation vector. Its Lie algebra $\mathfrak{se}(3)$ is the vector space of 4×4 twists $\hat{\xi}$ parameterized by the *twist coordinates* $\xi \in \mathbb{R}^6$, with the mapping [1]

$$\xi \stackrel{\Delta}{=} \begin{bmatrix} \omega \\ v \end{bmatrix} \rightarrow \hat{\xi} \stackrel{\Delta}{=} \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix}$$

Note we follow Frank Park's convention and reserve the first three components for rotation, and the last three for translation. Hence, with this parameterization, the generators for $SE(3)$ are

$$G^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G^5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G^6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Applying the exponential map to a twist ξ yields a screw motion yielding an element in $SE(3)$:

$$T = \exp \hat{\xi}$$

A closed form solution for the exponential map is given in [1, page 42].

5.1 The Adjoint Map

The adjoint is

$$\begin{aligned}
Ad_T\hat{\xi} &= T\hat{\xi}T^{-1} \\
&= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} [R\omega]_\times & -[R\omega]_\times t + Rv \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} [R\omega]_\times & t \times R\omega + Rv \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

From this we can express the Adjoint map in terms of twist coordinates (see also [1] and FP):

$$\begin{bmatrix} \omega' \\ v' \end{bmatrix} = \begin{bmatrix} R & 0 \\ [t]_\times R & R \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix}$$

5.2 Derivatives of Mappings

Hence, as with $SO(3)$, we are now in a position to simply posit the derivative of **inverse**,

$$\frac{\partial T^{-1}}{\partial \xi} = -\begin{bmatrix} R & 0 \\ [t]_\times R & R \end{bmatrix}$$

(but unit test on the above fails !!!), **compose** in its first argument,

$$\frac{\partial (T_1 T_2)}{\partial \xi_1} = \begin{bmatrix} R_2^T & 0 \\ [-R_2^T t]_\times R_2^T & R_2^T \end{bmatrix}$$

compose in its second argument,

$$\frac{\partial (T_1 T_2)}{\partial \xi_2} = I_6$$

between in its first argument,

$$\frac{\partial \left(T_1^{-1} T_2\right)}{\partial \xi_1} = -\begin{bmatrix} R & 0 \\ [t]_\times R & R \end{bmatrix}$$

with

$$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} = T_1^{-1} T_2 = between(T_2, T_1)$$

and between in its second argument,

$$\frac{\partial \left(T_1^{-1} T_2\right)}{\partial \xi_1} = I_6$$

5.3 The derivatives of Actions

The action of $SE(3)$ on 3D points is done by embedding the points in $\mathbb{R}^4$ by using homogeneous coordinates

$$\hat{q} = \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = T\hat{p}$$

We would now like to know what an incremental rotation parameterized by ξ would do:

$$\hat{q}(\xi) = Te^{\hat{\xi}}\hat{p}$$

hence the derivative (following the exposition in Section 3):

$$\frac{\partial \hat{q}(\xi)}{\partial \xi} = T\frac{\partial}{\partial \xi}\left(\hat{\xi}\hat{p}\right) = TH_p$$

where $\hat{\xi}\hat{p}$ corresponds to a velocity in $\mathbb{R}^4$ (in the local T frame):

$$\hat{\xi}\hat{p} = \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} \omega \times p + v \\ 0 \end{bmatrix}$$

Notice how velocities are anologous to points at infinity in projective geometry: they correspond to free vectors indicating a direction and magnitude of change.

By only taking the top three rows, we can write this as a velocity in $\mathbb{R}^3$, as the product of a 3×6 matrix H_p that acts upon the exponential coordinates ξ directly:

$$\omega \times p + v = -p \times \omega + v = \begin{bmatrix} -[p]_\times & I_3 \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} = H_p\xi$$

Hence, the final derivative of the group action is

$$\frac{\partial \hat{q}(\xi)}{\partial \xi} = T\hat{H}_p = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [-p]_\times & I_3 \\ 0 & 0 \end{bmatrix}$$

in homogenous coordinates. In $\mathbb{R}^3$ this becomes:

$$\frac{\partial q(\xi)}{\partial \xi} = R\begin{bmatrix} -[p]_\times & I_3 \end{bmatrix}$$

6 2D Rotations

The Lie group $SO(2)$ is a subgroup of the general linear group $GL(2)$ of 2×2 invertible matrices. Its Lie algebra $\mathfrak{so}(2)$ is the vector space of 2×2 skew-symmetric matrices. Though simpler than $SO(3)$ it is *commutative* and hence things simplify in ways that do not generalize well, so we treat it only now. Since $SO(2)$ is a one-dimensional manifold, $\mathfrak{so}(2)$ is isomorphic to $\mathbb{R}$ and we define

$$\hat{} : \mathbb{R} \to \mathfrak{so}(2)$$

$$\hat{} : \theta \to \hat{\theta} = [\theta]_+$$

which maps the angle θ to the 2×2 skew-symmetric matrix $[\theta]_+$:

$$[\theta]_+ = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$$

The exponential map can be computed in closed form as

$$R = e^{[\theta]_+} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

6.1 Derivatives of Mappings

The adjoint map for $\mathfrak{so}(2)$ is trivially equal to the identity, as is the case for *all* commutative groups, and we have the derivative of **inverse**,

$$\frac{\partial R^T}{\partial \theta} = -Ad_R = -1$$

compose in its first argument,

$$\frac{\partial (R_1 R_2)}{\partial \theta_1} = Ad_{R_2^T} = 1$$

compose in its second argument,

$$\frac{\partial (R_1 R_2)}{\partial \theta_2} = 1$$

between in its first argument,

$$\frac{\partial \left(R_1^T R_2\right)}{\partial \theta_1} = -Ad_{R_2^T R_1} = -1$$

and between in its second argument,

$$\frac{\partial \left(R_1^T R_2\right)}{\partial \theta_2} = 1$$

6.2 Derivatives of Actions

In the case of $SO(2)$ the vector space is $\mathbb{R}^2$, and the group action corresponds to rotating a point

$$q = Rp$$

We would now like to know what an incremental rotation parameterized by θ would do:

$$q(\theta) = Re^{[\theta]_+} p$$

hence the derivative (following the exposition in Section 3):

$$\frac{\partial q(\omega)}{\partial \omega} = R \frac{\partial}{\partial \omega} \left(e^{[\theta]_+} p \right) = R \frac{\partial}{\partial \omega} ([\theta]_+ p) = RH_p$$

Note that

$$[\theta]_+ \left[\begin{array}{c} x \\ y \end{array} \right] = \theta R_{\pi/2} \left[\begin{array}{c} x \\ y \end{array} \right] = \theta \left[\begin{array}{c} -y \\ x \end{array} \right] \tag{7}$$

which acts like a restricted "cross product" in the plane. Hence

$$[\theta]_+ p = \left[\begin{array}{c} -y \\ x \end{array} \right] \theta = H_p \theta$$

with $H_p = R_{pi/2} p$. Hence, the final derivative of an action in its first argument is

$$\frac{\partial q(\theta)}{\partial \theta} = RH_p = RR_{pi/2} p = R_{pi/2} Rp = R_{pi/2} q$$

7 2D Rigid Transformations

The Lie group $SE(2)$ is a subgroup of the general linear group $GL(3)$ of 3×3 invertible matrices of the form

$$T \stackrel{\Delta}{=} \left[\begin{array}{cc} R & t \\ 0 & 1 \end{array} \right]$$

where $R \in SO(2)$ is a rotation matrix and $t \in \mathbb{R}^2$ is a translation vector. Its Lie algebra $\mathfrak{se}(2)$ is the vector space of 3×3 twists $\hat{\xi}$ parameterized by the *twist coordinates* $\xi \in \mathbb{R}^3$, with the mapping

$$\xi \stackrel{\Delta}{=} \left[\begin{array}{c} v \\ \omega \end{array} \right] \rightarrow \hat{\xi} \stackrel{\Delta}{=} \left[\begin{array}{cc} [\omega]_+ & v \\ 0 & 0 \end{array} \right]$$

Note we think of robots as having a pose (x, y, θ) and hence I switched the order above, reserving the first two components for translation and the last for rotation. The Lie group generators are

$$G^x = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad G^y = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right] \quad G^\theta = \left[\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Applying the exponential map to a twist ξ yields a screw motion yielding an element in $SE(2)$:

$$T = \exp \hat{\xi}$$

A closed form solution for the exponential map is in the works...

7.1 The Adjoint Map

The adjoint is

$$\begin{aligned}
Ad_T \hat{\xi} &= T \hat{\xi} T^{-1} \\
&= \left[\begin{array}{cc} R & t \\ 0 & 1 \end{array} \right] \left[\begin{array}{cc} [\omega]_+ & v \\ 0 & 0 \end{array} \right] \left[\begin{array}{cc} R^T & -R^T t \\ 0 & 1 \end{array} \right] \\
&= \left[\begin{array}{cc} [\omega]_+ & -[\omega]_+ t + Rv \\ 0 & 0 \end{array} \right] \\
&= \left[\begin{array}{cc} [\omega]_+ & Rv - \omega R_{\pi/2} t \\ 0 & 0 \end{array} \right]
\end{aligned}$$

From this we can express the Adjoint map in terms of plane twist coordinates:

$$\begin{bmatrix} v' \\ \omega' \end{bmatrix} = \begin{bmatrix} R & -R_{\pi/2}t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

7.2 Derivatives of Mappings

We can just define all derivatives in terms of the above adjoint map:

$$\frac{\partial T^{-1}}{\partial \xi} = -Ad_T$$

$$\frac{\partial (T_1 T_2)}{\partial \xi_1} = Ad_{T_2^{-1}} = 1 \text{ and } \frac{\partial (T_1 T_2)}{\partial \xi_2} = I_3$$

$$\frac{\partial \left(T_1^{-1} T_2\right)}{\partial \xi_1} = -Ad_{T_2^{-1} T_1} = -Ad_{between(T_2, T_1)} \text{ and } \frac{\partial \left(T_1^{-1} T_2\right)}{\partial \xi_2} = I_3$$

7.3 The derivatives of Actions

The action of $SE(2)$ on 2D points is done by embedding the points in $\mathbb{R}^3$ by using homogeneous coordinates

$$\hat{q} = \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = T\hat{p}$$

Analoguous to $SE(3)$, we can compute a velocity $\hat{\xi}\hat{p}$ in the local T frame:

$$\hat{\xi}\hat{p} = \begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} [\omega]_+ p + v \\ 0 \end{bmatrix}$$

By only taking the top two rows, we can write this as a velocity in $\mathbb{R}^2$, as the product of a 2×3 matrix H_p that acts upon the exponential coordinates ξ directly:

$$[\omega]_+ p + v = v + R_{pi/2} p \omega = \begin{bmatrix} I_2 & R_{pi/2} p \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} = H_p \xi$$

Hence, the final derivative of the group action is

$$\frac{\partial q(\xi)}{\partial \xi} = R \begin{bmatrix} I_2 & R_{pi/2} p \end{bmatrix} = \begin{bmatrix} R & R_{pi/2} q \end{bmatrix}$$

8 Rot2 (in gtsam)

A rotation is stored as $(\cos\theta, \sin\theta)$. An incremental rotation is applied using the trigonometric sum rule:

$$\cos\theta' = \cos\theta\cos\delta - \sin\theta\sin\delta$$
$$\sin\theta' = \sin\theta\cos\delta + \cos\theta\sin\delta$$

where δ is an incremental rotation angle.

Derivatives of unrotate

$$\begin{aligned}
\frac{\partial x'}{\partial\delta} &= \frac{\partial(x\cos\theta' + y\sin\theta')}{\partial\delta} \\
&= \frac{\partial(x(\cos\theta\cos\delta - \sin\theta\sin\delta) + y(\sin\theta\cos\delta + \cos\theta\sin\delta))}{\partial\delta} \\
&= x(-\cos\theta\sin\delta - \sin\theta\cos\delta) + y(-\sin\theta\sin\delta + \cos\theta\cos\delta) \\
&= -x\sin\theta + y\cos\theta = y'
\end{aligned}$$

$$\begin{aligned}
\frac{\partial y'}{\partial\delta} &= \frac{\partial(-x\sin\theta' + y\cos\theta')}{\partial\delta} \\
&= \frac{\partial(-x(\sin\theta\cos\delta + \cos\theta\sin\delta) + y(\cos\theta\cos\delta - \sin\theta\sin\delta))}{\partial\delta} \\
&= -x(-\sin\theta\sin\delta + \cos\theta\cos\delta) + y(-\cos\theta\sin\delta - \sin\theta\cos\delta) \\
&= -x\cos\theta - y\sin\theta = -x'
\end{aligned}$$

$$\frac{\partial p'}{\partial p} = \frac{\partial(Rp)}{\partial p} = R$$

9 Point3

A cross product $a \times b$ can be written as a matrix multiplication

$$a \times b = [a]_\times b$$

where $[a]_\times$ is a skew-symmetric matrix defined as

$$[x, y, z]_\times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

We also have

$$a^T[b]_\times = -([b]_\times a)^T = -(a \times b)^T$$

The derivative of a cross product

$$\frac{\partial(a \times b)}{\partial a} = [-b]_\times \tag{8}$$

$$\frac{\partial(a \times b)}{\partial b} = [a]_\times \tag{9}$$

10 Pose3 (gtsam, old-style exmap)

In the old-style, we have

$R' = R(I+\Omega)$

$t' = t + dt$

In this case, the derivative of ***transform_from***, $Rx+t$:

$$\frac{\partial(R(I+\Omega)x+t)}{\partial\omega} = \frac{\partial(R\Omega x)}{\partial\omega} = \frac{\partial(R(\omega\times x))}{\partial\omega} = R[-x]_\times$$

and with respect to dt is easy:

$$\frac{\partial(Rx+t+dt)}{\partial dt} = I$$

The derivative of ***transform_to***, $inv(R)(x-t)$ we can obtain using the chain rule:

$$\frac{\partial(inv(R)(x-t))}{\partial\omega} = \frac{\partial unrot(R,(x-t))}{\partial\omega} = skew(R^T(x-t))$$

and with respect to dt is easy:

$$\frac{\partial(R^T(x-t-dt))}{\partial dt} = -R^T$$

11 Pose3 (gtsam, new-style exmap)

In the new-style exponential map, Pose3 is composed with a delta pose as follows

$R' = (I+\Omega)R$

$t' = (I+\Omega)t + dt$

The derivative of transform_from, $Rx+t$:

$$\frac{\partial((I+\Omega)Rx+(I+\Omega)t)}{\partial\omega} = \frac{\partial(\Omega(Rx+t))}{\partial\omega} = \frac{\partial(\omega\times(Rx+t))}{\partial\omega} = -[Rx+t]_\times$$

and with respect to dt is easy:

$$\frac{\partial(Rx+t+dt)}{\partial dt} = I$$

The derivative of transform_to, $R^T(x-t)$, eludes me. The calculation below is just an attempt:

Noting that $R'^T = R^T(I-\Omega)$, and $(I-\Omega)(x-(I+\Omega)t) = (I-\Omega)(x-t-\Omega t) = x-t-dt-\Omega x+\Omega^2 t$

$$\frac{\partial(R'^T(x-t'))}{\partial\omega} = \frac{\partial(R^T(I-\Omega)(x-(I+\Omega)t))}{\partial\omega} = -\frac{\partial(R^T(\Omega(x-\Omega t)))}{\partial\omega}$$

$$-\frac{\partial([R^T\omega]_\times R^T x)}{\partial\omega} = [R^T x]_\times \frac{\partial(R^T\omega)}{\partial\omega} = [R^T x]_\times R^T$$

$$= \frac{\partial(R^T\Omega^2 t)}{\partial\omega} + [R^T x]_\times R^T$$

and with respect to dt is easy:

$$\frac{\partial(R^T(x-t-dt))}{\partial dt} = -R^T$$

12 Line3vd

One representation of a line is through 2 vectors (v,d), where v is the direction and the vector d points from the orgin to the closest point on the line.

In this representation, transforming a 3D line from a world coordinate frame to a camera at (R_w^c, t^w) is done by

$$\begin{aligned} v^c &= R_w^c v^w \\ d^c &= R_w^c (d^w + (t^w v^w) v^w - t^w) \end{aligned}$$

13 Line3

For 3D lines, we use a parameterization due to C.J. Taylor, using a rotation matrix R and 2 scalars a and b. The line direction v is simply the Z-axis of the rotated frame, i.e., $v = R_3$, while the vector d is given by $d = aR_1 + bR_2$.

Now, we will *not* use the incremental rotation scheme we used for rotations: because the matrix R translates from the line coordinate frame to the world frame, we need to apply the incremental rotation on the right-side:

$$R' = R(I + \Omega)$$

Projecting a line to 2D can be done easily, as both v and d are also the 2D homogenous coordinates of two points on the projected line, and hence we have

$$\begin{aligned} l &= v \times d \\ &= R_3 \times (aR_1 + bR_2) \\ &= a(R_3 \times R_1) + b(R_3 \times R_2) \\ &= aR_2 - bR_1 \end{aligned}$$

This can be written as a rotation of a point,

$$l = R \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$$

but because the incremental rotation is now done on the right, we need to figure out the derivatives again:

$$\frac{\partial(R(I+\Omega)x)}{\partial \omega} = \frac{\partial(R\Omega x)}{\partial \omega} = R\frac{\partial(\Omega x)}{\partial \omega} = R[-x]_\times \tag{10}$$

and hence the derivative of the projection l with respect to the rotation matrix $R$of the 3D line is

$$\frac{\partial(l)}{\partial \omega} = R[\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}]_\times = \begin{bmatrix} aR_3 & bR_3 & -(aR_1 + bR_2) \end{bmatrix} \tag{11}$$

or the a, b scalars:

$$\frac{\partial(l)}{\partial a} = R_2$$
$$\frac{\partial(l)}{\partial b} = -R_1$$

Transforming a 3D line $(R, (a, b))$ from a world coordinate frame to a camera frame (R^c_w, t^w) is done by

$$R' = R^c_w R$$
$$a' = a - R_1^T t^w$$
$$b' = b - R_2^T t^w$$

Again, we need to redo the derivatives, as R is incremented from the right. The first argument is incremented from the left, but the result is incremented on the right:

$$\begin{aligned}
R'(I+\Omega') = (AB)(I+\Omega') &= (I+[S\omega]_\times)AB \\
I+\Omega' &= (AB)^T(I+[S\omega]_\times)(AB) \\
\Omega' &= R'^T[S\omega]_\times R' \\
\Omega' &= [R'^T S\omega]_\times \\
\omega' &= R'^T S\omega
\end{aligned}$$

For the second argument R we now simply have:

$$\begin{aligned}
AB(I+\Omega') &= AB(I+\Omega) \\
\Omega' &= \Omega \\
\omega' &= \omega
\end{aligned}$$

The scalar derivatives can be found by realizing that

$$\begin{pmatrix} a' \\ b' \\ \dots \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} - R^T t^w$$

where we don't care about the third row. Hence

$$\frac{\partial((R(I+\Omega_2))^T t^w)}{\partial\omega} = -\frac{\partial(\Omega_2 R^T t^w)}{\partial\omega} = -[R^T t^w]_\times = \begin{bmatrix} 0 & R_3^T t^w & -R_2^T t^w \\ -R_3^T t^w & 0 & R_1^T t^w \\ \dots & \dots & 0 \end{bmatrix}$$

14 2D Line Segments

The error between an infinite line (a, b, c) and a 2D line segment $((x1, y1), (x2, y2))$ is defined in Line3.ml.

15 Recovering Pose

Below is the explanaition underlying Pose3.align, i.e. aligning two point clouds using SVD. Inspired but modified from CVOnline...

Our model is

$$p^c = R\left(p^w - t\right)$$

i.e., R is from camera to world, and t is the camera location in world coordinates. The objective function is

$$\frac{1}{2}\sum\left(p^c - R(p^w - t)\right)^2 = \frac{1}{2}\sum\left(p^c - Rp^w + Rt\right)^2 = \frac{1}{2}\sum\left(p^c - Rp^w - t'\right)^2 \tag{12}$$

where $t' = -Rt$ is the location of the origin in the camera frame. Taking the derivative with respect to t' and setting to zero we have

$$\sum\left(p^c - Rp^w - t'\right) = 0$$

or

$$t' = \frac{1}{n}\sum\left(p^c - Rp^w\right) = \bar{p}^c - R\bar{p}^w \tag{13}$$

here $\bar{p}^c$ and $\bar{p}^w$ are the point cloud centroids. Substituting back into (12), we get

$$\frac{1}{2}\sum\left(p^c - R(p^w - t)\right)^2 = \frac{1}{2}\sum\left((p^c - \bar{p}^c) - R\left(p^w - \bar{p}^w\right)\right)^2 = \frac{1}{2}\sum\left(\hat{p}^c - R\hat{p}^w\right)^2$$

Now, to minimize the above it suffices to maximize (see CVOnline)

$$trace\left(R^T C\right)$$

where $C = \sum \hat{p}^c \left(\hat{p}^w\right)^T$ is the correlation matrix. Intuitively, the cloud of points is rotated to align with the principal axes. This can be achieved by SVD decomposition on C

$$C = USV^T$$

and setting

$$R = UV^T$$

Clearly, from (13) we then also recover the optimal t as

$$t = \bar{p}^w - R^T \bar{p}^c$$

References

[1] R.M. Murray, Z. Li, and S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.