Manifold Geometry of the Sphere S^2

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Retraction

Suppose we have a point $p \in S^2$ and a 3-vector z, Absil [1] tells us we can simply add z to p and renormalize to get a new point q on the sphere. This is what he calls a **retraction** $R_p(z)$,

$$q = R_p(z) = \frac{p+z}{\|p+z\|} = \frac{p+z}{\alpha}$$

with α the norm of $p+z$. The only restriction on z is that it is in the tangent space T_pS^2 at p, i.e., $p^Tz = 0$. Multiplying with p^T on both sides we have

$$\alpha p^T q = p^T p + p^T z$$

and (since $p^Tp = 1$ and $p^Tz = 0$) we have $\alpha = 1/(p^Tq)$.

Inverse

Suppose we are given points p and q on the sphere, what is the tangent vector z that takes p to q? We can find a basis B for the tangent space, with $B = [b_1|b_2]$ a 3×2 matrix, by either

1. Decompose $p = QR$, with Q orthonormal and R of the form $[1\,0\,0]^T$, and hence $p = Q_1$. The basis $B = [Q_2|Q_3]$, i.e., the last two columns of Q.

2. Form $b_1 = p \times a$, with a (consistently) chosen to be non-parallel to p, and $b_2 = p \times b_1$.

Now, if $z = B\xi$ with $\xi \in R^2$ the 2D coordinate in the tangent plane basis B, we have

$$\alpha q = p + z = p + B\xi$$

If we multiply both sides with B^T (project on the basis B) we obtain

$$\alpha B^T q = B^T p + B^T B\xi$$

and because $B^Tp = 0$ and $B^TB = I$ we trivially obtain ξ as the scaled projection B^Tq:

$$\xi = \alpha B^T q = \frac{B^T q}{p^T q}$$

Exponential Map

The exponential map itself is not so difficult, and is given in Ma01ijcv, as well as in this CVPR tutorial by Anuj Srivastava: http://stat.fsu.edu/∼anuj/CVPR_Tutorial/Part2.pdf.

$$\exp_p z = \cos\left(\|z\|\right) p + \sin\left(\|z\|\right) \frac{z}{\|z\|}$$

The latter also gives the inverse, i.e., get the tangent vector z to go from p to q:

$$z = \log_p q = \frac{\theta}{\sin\theta}\left(q - p\cos\theta\right) p$$

with $\theta = \cos^{-1}\left(p^T q\right)$.

References

[1] P.-A. Absil, R. Mahony, and R. Sepulchre. *Optimization Algorithms on Matrix Manifolds*. Princeton University Press, Princeton, NJ, USA, 2007.